

Actions of $GL_q(2, C)$ on $C(1, 3)$ and its four dimensional representations

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Abstract. A complete classification is given of all inner actions on the Clifford algebra $C(1, 3)$ defined by representations of the quantum group $GL_q(2, C)$, $q^m \neq 1$ with nonzero perturbations. As a consequence of this classification it is shown that the space of invariants of every $GL_q(2, C)$ -action of this type, which is not an action of $SL_q(2, C)$, is generated by 1 and the value of the quantum determinant for the given representation.

1 Introduction

The development of the quantum group theory and its applications in mathematical physics (more exactly an interpretation of some results using physical intuitions) give us the hope that the notions of quantum actions, quantum invariants and quantum symmetries could play an important role in the “quantum” mathematics as the classical notions of invariants and symmetry do in classical mathematics and theoretical physics. In this paper we study in details actions and invariants of the quantum group $GL_q(2, C)$ on the space-time Clifford algebra $C(1, 3)$, which is generated by vectors $\gamma_\mu, \mu = 0, 1, 2, 3$ with relations defined by the form $g_{\mu\nu} = (1, -1, -1, -1)$. This algebra has 16

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matrix units

$$\begin{aligned}
e_{11} &= (1 + \gamma_0)(1 + i\gamma_{12})/4, & e_{21} &= (1 + \gamma_0)(i\gamma_2 - \gamma_1)\gamma_3/4, \\
e_{31} &= (1 - \gamma_0)(1 + i\gamma_{12})\gamma_3/4, & e_{41} &= (1 - \gamma_0)(\gamma_1 - i\gamma_2)/4, \\
e_{12} &= (1 + \gamma_0)(\gamma_1 + i\gamma_2)\gamma_3/4, & e_{22} &= (1 + \gamma_0)(1 - i\gamma_{12})/4, \\
e_{32} &= (1 - \gamma_0)(\gamma_1 + i\gamma_2)/4, & e_{42} &= (1 - \gamma_0)(i\gamma_{12} - 1)\gamma_3/4, \\
e_{13} &= -(1 + \gamma_0)(1 + i\gamma_{12})\gamma_3/4, & e_{23} &= -(1 + \gamma_0)(\gamma_1 - i\gamma_2)/4, \\
e_{33} &= (1 - \gamma_0)(1 + i\gamma_{12})/4, & e_{43} &= (1 - \gamma_0)(-\gamma_1 + i\gamma_2)\gamma_3/4, \\
e_{14} &= -(1 + \gamma_0)(\gamma_1 + i\gamma_2)/4, & e_{24} &= (1 + \gamma_0)(1 - i\gamma_{12})\gamma_3/4, \\
e_{34} &= (1 - \gamma_0)(\gamma_1 + i\gamma_2)\gamma_3/4, & e_{44} &= (1 - \gamma_0)(1 - i\gamma_{12})/4.
\end{aligned} \tag{1}$$

and therefore it is abstractly isomorphic to the algebra of 4 by 4 complex matrices.

The quantum group $GL_q(2, C)$ is generated by four, so-called, q -spinors; therefore the classification of representations of the q -spinor in $C(1, 3)$ given in Theorem t2 produces a basic tool for the investigation of the inner actions defined by representations of the algebraic structure of $GL_q(2, C)$. In the fourth section we find a necessary and sufficient condition for different representations to define equal or equivalent actions. Using this result in Sections 6, 7 we have obtained a complete classification of all inner actions defined by representations with nonzero “perturbations” (i.e. when two main q -spinors do not commute). Results are summarized in the Table given in Section 8. As a consequence of this classification we have shown that the space of quantum invariants of every $GL_q(2, C)$ -action of this type, which is not an action of $SL_q(2, C)$, is generated by 1 and the value of the quantum determinant. Roughly speaking, it means that in this case the quantum determinants are the only quantum invariants.

We believe that the results of this classification, besides possible interpretations with physical intuition, could be useful as a starting material for investigations of actions of more complicated quantum groups on another natural classical objects.

2 Preliminary notions

The quantum group $GL_q(2, C)$ is a Hopf algebra whose algebraic structure is generated by five elements $a_{11}, a_{12}, a_{21}, a_{22}, d^{-1}$ which can be presented by the following diagram:

$$\begin{array}{ccc}
 a_{11} & \longrightarrow & a_{12} \\
 \downarrow & \nearrow \cdot & \downarrow \\
 & \cdot & \\
 a_{21} & \longrightarrow & a_{22}
 \end{array}
 \quad (2)$$

where by the arrows $x \rightarrow y$ are denoted the so-called q -spinors $xy = qyx$, by the straight line commuting elements and by dots, elements with nontrivial commutator $[a_{11}a_{22}] = (q - q^{-1})a_{12}a_{21}$ (for another quantum deformations of GL_n see [1],[3] [4], [14]). The comultiplication and the counit are defined as follows: $\Delta(a_{ij}) = \sum_{k=1}^{k=2} a_{ik} \otimes a_{kj}$, $\varepsilon(a_{ij}) = \delta_i^j$. The antipode is given by the following formula

$$S \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = d^{-1} \begin{pmatrix} a_{22} & -q^{-1}a_{12} \\ -qa_{21} & a_{11} \end{pmatrix}. \quad (3)$$

The quantum group $SL_q(2, C)$ is defined as the factor-Hopf algebra of $GL_q(2, C)$ by the additional relation $d = 1$.

An action of Hopf algebra H on an algebra A is characterized by the following two main formulae

$$(hg) \cdot v = h \cdot (g \cdot v), \quad (4)$$

$$h \cdot vw = \sum (h_{(1)} \cdot v)(h_{(2)} \cdot w), \quad (5)$$

where $h, g \in H$, $v, w \in A$ and $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ (see details in [2], [13]).

The first formula shows that the action will be defined if it is defined for generators a_{ij} of $GL_q(2, C)$, while the second is showing that for a definition of an action it is enough to set the action of a_{ij} on the generators $\gamma_0, \gamma_1, \gamma_2$,

γ_3 of the Clifford algebra $C(1, 3)$. Thus, an action is set by formulas of the following type

$$a_{ij} \cdot \gamma_k = f_{ijk}(\gamma_0, \gamma_1, \gamma_2, \gamma_3) \quad (6)$$

where f_{ijk} are some noncommutative polynomials in four variables.

Two actions $*$ and \cdot of a Hopf algebra H on an algebra A are called *equivalent* if $h * v = (h \cdot v \zeta^{-1})^\zeta$, where ζ is an automorphism of the algebra A . By Skolem—Noether theorem every automorphism of $C(1, 3)$ is given by a conjugation $w^\zeta = u w u^{-1}$. Therefore equivalence of actions can be presented by the following formula

$$a_{ij} * (u w u^{-1}) = u(a_{ij} \cdot w) u^{-1}. \quad (7)$$

This formula shows that for an action $*$ to be equivalent to the action \cdot there exists a presentation

$$a_{ij} * \gamma'_k = f_{ijk}(\gamma'_0, \gamma'_1, \gamma'_2, \gamma'_3) \quad (8)$$

with the same polynomials f_{ijk} and a system of generators $\gamma'_0, \gamma'_1, \gamma'_2, \gamma'_3$ with the same relations.

Using Skolem—Noether theorem (see also [5], or [9] chapter 6.2.) it is easy to show that for every action there exists an invertible 2 by 2 matrix M over $C(1, 3)$ such that

$$\begin{pmatrix} a_{11} \cdot v & a_{12} \cdot v \\ a_{21} \cdot v & a_{22} \cdot v \end{pmatrix} = M \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} M^{-1}. \quad (9)$$

Inversely, if $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ is a 2 by 2 matrix with the additional condition that $a_{ij} \rightarrow A_{ij} = m_{ij}$ represents a homomorphism of the algebraic structure of $GL_q(2, C)$ to $C(1, 3)$, then M is invertible and (sko) defines an action of $GL_q(2, C)$ on $C(1, 3)$ (called an *inner* action).

For a given representation $a_{ij} \rightarrow A_{ij}$ we denote by \mathfrak{R} an *operator algebra* i.e. a subalgebra of $C(1, 3)$ generated by A_{ij} . Recall that the algebra of invariants of an action is defined in the following way

$$Inv = \{v \in A | \forall h \in H \quad h \cdot v = \varepsilon(h)v\}. \quad (10)$$

Lemma 1 . *If an action of $GL_q(2, C)$ is given by the formula (sko) then the algebra Inv equals the centralizer of all components of M . In particular the algebra Inv for an inner action on $C(1, 3)$ defined by a representation $a_{ij} \rightarrow A_{ij}$ equals the centralizer of \mathfrak{R} in $C(1, 3)$.*

Proof. An element v is an invariant if and only if $a_{11} \cdot v = \varepsilon(a_{11})v = v$, $a_{12} \cdot v = \varepsilon(a_{12})v = 0$, $a_{21} \cdot v = \varepsilon(a_{21})v = 0$, $a_{22} \cdot v = \varepsilon(a_{22})v = v$. In matrix form this is equivalent to $diag(v, v) = M diag(v, v)M^{-1}$. Therefore $diag(v, v)M = M diag(v, v)$, i.e. $vm_{ij} = m_{ij}v$ for all components m_{ij} of M . \square

3 q -spinor representations

Let $(x, y) \in A_q^{2/0}$ be a q -spinor $xy = qyx$. If $x \rightarrow A$, $y \rightarrow B$ is its representation by 4×4 matrices over complex numbers, then for every invertible 4×4 matrix u and nonzero number α , the map $x \rightarrow uAu^{-1}\alpha$, $y \rightarrow uBu^{-1}\alpha$ also defines a representation of the q -spinor. We consider these two representations as *equivalent* ones. Thus, under investigation of representations of a q -spinor, we can suppose that the matrix A has a Jordan Normal form and one of its eigenvalues is equal to 1 (if $A \neq 0$).

For a given matrix A , a set $B(A)$ of all matrices B , such that $x \rightarrow A$, $y \rightarrow B$ is a representation of the q -spinor, forms a linear space. Therefore, it is natural to represent the space $B(A)$ by one of its basis: $\{B_1, B_2, \dots\}$.

Theorem 1 . *Every representation of the q -spinor ($q^3, q^4 \neq 1$) by 4×4 complex matrices, $x \rightarrow A$, $y \rightarrow B$, such that A is invertible and $B(A)^2 \neq 0$, is equivalent to one of the following representations*

$$1. A = diag(q^2, q, 1, 1), \quad B_1 = e_{12}, \quad B_2 = e_{23}, \quad B_3 = e_{24} \quad (11)$$

$$2. A = diag(q^2, q, q, 1), \quad B_1 = e_{12}, \quad B_2 = e_{13}, \quad B_3 = e_{24}, \quad B_4 = e_{34} \quad (12)$$

$$3. A = diag(q^2, q^2, q, 1), \quad B_1 = e_{13}, \quad B_2 = e_{23}, \quad B_3 = e_{34} \quad (13)$$

$$4. A = diag(q^3, q^2, q, 1), \quad B_1 = e_{12}, \quad B_2 = e_{23}, \quad B_3 = e_{34} \quad (14)$$

$$5. A = \text{diag}(\alpha, q^2, q, 1), \quad B_1 = e_{23}, \quad B_2 = e_{34}, \quad \alpha \neq 0, q^{-1}, 1, q, q^2, q^3 \quad (15)$$

$$6. A = \text{diag}(q^2, q^2, q, 1) + e_{12}, \quad B_1 = e_{13}, \quad B_2 = e_{34} \quad (16)$$

$$7. A = \text{diag}(q^2, q, 1, 1) + e_{34}, \quad B_1 = e_{24}, \quad B_2 = e_{12} \quad (17)$$

Lemma 2 . Let a matrix A have the form $A = \alpha E + U$, where $\alpha \neq 0$, U is a nilpotent $n \times n$ -matrix, E is the identity matrix, $E = \text{diag}(1, 1, \dots, 1)$. If $AB = qBA$, $q \neq 1$, then $B = 0$.

Proof. If we denote $[x, y]_q = xy - qyx$ then $[A, B]_q = 0$; i.e.

$$[\alpha E + U, B]_q = \alpha(1 - q)B + [U, B]_q. \quad (18)$$

This formula implies $[U, B]_q = \alpha(q - 1)B$. Iterating this equality m times we have

$$[U, [U, [\dots [U, B]_q \dots]_q]_q = \alpha^m (q - 1)^m B. \quad (19)$$

If $U^k = 0$ and $m = 2k$ then the left hand part of (f42) is equal to zero. Thus $\alpha^m (q - 1)^m B = 0$ and $B = 0$. \square

Lemma 3 . If $x \rightarrow A$, $y \rightarrow B$ is a representation of the q -spinor and C_1, C_2 are matrices commuting with A , then $x \rightarrow A$, $y \rightarrow C_1 B C_2$ is also a representation of the q -spinor.

Proof. $A \cdot C_1 B C_2 = C_1 A \cdot B C_2 = C_1 q B A \cdot C_2 = q C_1 B C_2 \cdot A$. \square

Lemma 4 . If A is a diagonal matrix then $B(A)$ has a basis consisting of matrix units.

Proof. If $B = ||\beta_{ij}|| \in B(A)$, then $\beta_{ij} e_{ij} = e_{ii} B e_{jj} \in B(A)$ (since $e_{ii} A = A e_{ii}$ for each diagonal matrix A and we can use Lemma 3). Thus, for every nonzero β_{ij} the matrix unit e_{ij} belongs to $B(A)$. \square

Proof of Theorem t2. We can assume that the matrix A has a Jordan Normal form and one of it's eigenvalues is equal to 1.

Assume firstly that A is a diagonal matrix; i.e. it has four simplest Jordan Normal blocks: $A = \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\alpha_4 = 1$. In this case, by Lemma 4, it is enough to describe all matrix units from $B(A)$. We have

$$Ae_{ij} = \alpha_i e_{ij} = \alpha_i \alpha_j^{-1} (\alpha_j e_{ij}) = \alpha_i \alpha_j^{-1} (e_{ij} A). \quad (20)$$

This formula shows that $e_{ij} \in B(A)$ if and only if $\alpha_i \alpha_j^{-1} = q$; i.e.

$$\alpha_i = q \alpha_j. \quad (21)$$

If $B(A)^2 \neq 0$, then there exist indices i, j, k such that $e_{ij}, e_{jk} \in B(A)$. A conjugation by matrix $T = E - e_{ii} - e_{jj} + e_{ij} + e_{ji}$ changes i 'th and j 'th entries in any diagonal matrix. Therefore we can suppose that $i = 2, j = 3, k = 4$; i.e. (f44) implies $A = \text{diag}(\alpha, q^2, q, 1)$. If $\alpha \neq q^{-1}, 1, q, q^2, q^3$ then $B(A) = Ce_{23} + Ce_{34}$ and we have a representation (35). If $\alpha = q^{-1}, 1, q, q^2, q^3$ then we will obtain four possibilities which are equivalent to (f34), (f31), (f32), (f33) and (f34), respectively.

Now, let us consider the cases such that A has less than four blocks.

By Lemma 2, A cannot be a simplest Jordan Normal matrix; i.e. it has more than one block. This means that A has a form $A = \text{diag}(a, b)$, where a, b are either invertible 2×2 matrices in Jordan Normal form or a is an invertible simplest Normal Jordan 3×3 matrix and b is a nonzero complex number (and therefore we can suppose that $b = 1$). Let us write

$$B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0, \quad (22)$$

where $\alpha, \beta, \gamma, \delta$ are, respectively, either 2×2 matrices or α is a 3×3 matrix, δ is a complex number and β, γ , a column and a row, respectively. In both cases we have

$$[A, B]_q = \begin{pmatrix} a\alpha - q\alpha a & a\beta - q\beta b \\ b\gamma - q\gamma a & b\delta - q\delta b \end{pmatrix} = 0. \quad (23)$$

This implies, in particular, that $x \rightarrow a, y \rightarrow \alpha$ and $x \rightarrow b, y \rightarrow \delta$ are representations of q -spinors.

Let us consider firstly the case when a is a 3×3 matrix. In this case, by Lemma 2, we have $\alpha = 0$, $\delta = 0$ and also $a\beta = q\beta$, $\gamma = q\gamma a$ (recall that $b = 1$). Therefore for a 3×3 matrix $\beta\gamma$ (this is the matrix product of a column by a row), we have $a(\beta\gamma) = q\beta\gamma = q^2(\beta\gamma)a$. Again by Lemma 2 we obtain $\beta\gamma = 0$, therefore either $\beta = 0$ or $\gamma = 0$.

Let $\beta = 0$. Then $\gamma \neq 0$ and the equality $\gamma = q\gamma a$ can be written in the form $\gamma a = q^{-1}\gamma$. It means that q^{-1} is an eigenvalue of a and γ is an eigenvector of a ; i.e.

In this case $B(A) = Ce_{43}$ and $B(A)^2 = 0$. $a = q^{-1}E + e_{12} + e_{23}$, $\gamma = \epsilon e_{43}$. Let $\gamma = 0$, then $\beta \neq 0$ and the equality $a\beta = q\beta$ shows that q is the eigenvalue of a and β is an eigenvector i.e. $a = qE + e_{12} + e_{23}$, $\beta = \epsilon e_{14}$ and $B(A) = Ce_{14}$, so $B(A)^2 = 0$.

Consider now the case when $a, b, \alpha, \beta, \gamma, \delta$ are 2×2 matrices.

Let us start with the case when both matrices a and b have a simplest Jordan Normal Form i.e. $a = \epsilon E + e_{12}$, $b = E + e_{34}$. By formula (f47) and Lemma 2 we have $\alpha = \delta = 0$. If

$$B' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \quad (24)$$

is another matrix from $B(A)$; then we also have $\alpha' = \delta' = 0$ and by (f47) we have the relations $a\beta' = q\beta'b$, $b\gamma' = q\gamma'a$. This relations and (f47) imply $a(\beta\gamma') = q\beta b\gamma' = q^2(\beta\gamma')a$ and by Lemma 2, $\beta\gamma' = 0$.

In the same way $b(\gamma\beta') = q\gamma a\beta' = q^2(\gamma\beta')b$ and by Lemma 2, $\gamma\beta' = 0$. Now

$$BB' = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} 0 & \beta' \\ \gamma' & 0 \end{pmatrix} = \begin{pmatrix} \beta\gamma' & 0 \\ 0 & \gamma\beta' \end{pmatrix} = 0, \quad (25)$$

therefore $B(A)^2 = 0$.

Suppose now that one of the matrix a, b is a simplest Jordan matrix while another is a diagonal matrix. A conjugation by $T = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$, where E is the identity 2×2 matrix, $E = \text{diag}(1, 1)$, changes $A = \text{diag}(a, b)$ to $\text{diag}(b, a)$, so we can suppose that $a = \epsilon E + e_{12}$, $b = \text{diag}(\mu, 1)$, where $\epsilon, \mu \neq 0$.

By (f47) $x \rightarrow a$, $y \rightarrow \alpha$ is a representation of the q -spinor and Lemma 2 implies $\alpha = 0$. In the same way $x \rightarrow b$, $y \rightarrow \delta$ is a representation of the q -spinor. Therefore if $\mu \neq q$, q^{-1} then $\delta = 0$ and we can repeat word by word the proof beginning from formula (f54) up to (f58) in order to show that $B(A)^2 = 0$.

If $\mu = q^{-1}$ we can multiply matrices A and B by q and conjugate them by the matrix $\text{diag}(1, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$. We will obtain an equivalent representation with $\mu = q$. Thus, it is enough to consider the case

$$a = \epsilon E + e_{12}, \quad b = \text{diag}(q, 1) \quad (26)$$

and by Lemma 2

$$\alpha = 0, \quad \delta = c e_{12}, \quad c \in \mathbf{C}. \quad (27)$$

For $\beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$, by (f47), we have $\alpha\beta = q\beta b$, which implies

$$(\epsilon - q^2)\beta_{11} = -\beta_{21}, \quad (\epsilon - q)\beta_{12} = -\beta_{22} \quad (28)$$

$$(\epsilon - q^2)\beta_{21} = 0, \quad (\epsilon - q)\beta_{22} = 0. \quad (29)$$

If $\epsilon = q^2$ then the first equality of (f64) gives $\beta_{21} = 0$, and if $\epsilon \neq q^2$ then the first equality of (f65) gives $\beta_{21} = 0$. Therefore $\beta_{21} = 0$ in any case. In the same way $\beta_{22} = 0$ and (f64), (f65) are equivalent to

$$(\epsilon - q^2)\beta_{11} = 0 \quad (\epsilon - q)\beta_{12} = 0 \quad (30)$$

$$\beta_{21} = 0, \quad \beta_{22} = 0. \quad (31)$$

Analogously for the matrix $\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$ we have $b\gamma = q\gamma a$; which is equivalent to

$$\gamma_{11} = 0, \quad (1 - \epsilon)\gamma_{12} = 0, \quad (32)$$

$$\gamma_{21} = 0, \quad (1 - q\epsilon)\gamma_{22} = 0. \quad (33)$$

Now if $\epsilon \neq q^{-1}, 1, q, q^2$ then by (f66), (f67) and (f71), (f72) $\beta = \gamma = 0$ and the representation has the form

$$A = \text{diag} \left(\begin{pmatrix} \epsilon & 1 \\ 0 & \epsilon \end{pmatrix}, q, 1 \right), \quad B_1 = e_{34}. \quad (34)$$

This means $B(A) = Ce_{34}$ and $B(A)^2 = 0$.

Finally, let us consider four last possibilities.

1. $\epsilon = q^{-1}$. By (f66) and (f67) we have $\beta = 0$ and by (f71) and (f72), $\gamma = ce_{22}$. Together with (f61) and (f62) it follows

$$A = \text{diag} \left(\begin{pmatrix} q^{-1} & 1 \\ 0 & q^{-1} \end{pmatrix}, q, 1 \right), \quad B_1 = e_{42}, \quad B_2 = e_{34}. \quad (35)$$

If we multiply A by q and conjugate it by $T = \text{diag}(1, q^{-1}, 1, 1)$ we will obtain an equivalent representation $A = \text{diag}(1, 1, q^2, q) + e_{12}$, $B_1 = e_{42}$, $B_2 = e_{34}$. Using conjugations by matrices $E - e_{ii} - e_{jj} + e_{ij} + e_{ji}$ we can change indices with the help of permutation $1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 1, 4 \rightarrow 2$. In this case $e_{42} \rightarrow e_{24}$, $e_{34} \rightarrow e_{12}$ and we have the representation (f37).

2. $\epsilon = 1$. By (f66) and (f67) we again have $\beta = 0$ and by (f71) and (f72), $\gamma = Ce_{12}$. Now relations (f61) and (f62) show that the representation has the form

$$A = \text{diag} \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, q, 1 \right), \quad B_1 = e_{32} \quad B_2 = e_{34}. \quad (36)$$

and therefore $B(A)^2 = 0$.

3. $\epsilon = q$. By (f71) and (f72) we have $\gamma = 0$ and by (f66) and (f67) $\beta = ce_{12}$. With (f61) and (f62) this implies the representation has the form

$$A = \text{diag} \left(\begin{pmatrix} q & 1 \\ 0 & q \end{pmatrix}, q, 1 \right), \quad B_1 = e_{14}, \quad B_2 = e_{34} \quad (37)$$

and again $B(A)^2 = 0$.

4. $\epsilon = q^2$. By (f71) and (f72) we have $\gamma = 0$ and equalities (f66) and (f67) imply $\beta = ce_{11}$. So the representation has the form

$$A = \text{diag} \left(\begin{pmatrix} q^2 & 1 \\ 0 & q^2 \end{pmatrix}, q, 1 \right), \quad B_1 = e_{13}, \quad B_2 = e_{34}. \quad (38)$$

This representation coincides with (f36). \square

Note that if $q^3 = 1$ or $q^4 = 1$, then Theorem t2 is not valid. For $q = e^{\frac{2\pi i}{3}}$ and $q = \pm i$ there exist, respectively, three dimensional and four dimensional irreducible representations (see for instance [7]) while for $q^m \neq 1$ all finite dimensional representations of the q -spinor are one dimensional (see [11], Chapter 2).

4 Equivalence of representations

All irreducible finite dimensional representations of $GL_q(2, C)$, $q^m \neq 1$ are one dimensional. This is a folklore fact which can be easily obtained from the Y.S. Soibelman work [12] or from the FRT-duality [10] and the fact that $U_q(g)$ is pointed (see [8]). We need the following corollary from this fact.

Corollary 1 . *Every finite dimensional representation of $GL_q(2, C)$, $q^m \neq 1$, is triangular; i.e. it is equivalent to a representation by triangular matrices $a_{ij} \rightarrow A_{ij}$.*

Corollary 2 . *For every finite dimensional representation $a_{ij} \rightarrow A_{ij}$ of $GL_q(2, C)$, $q^m \neq 1$, the elements A_{11} , A_{22} are invertible, while A_{12} , A_{21} are nilpotent.*

Proof. From Corollary c2 we can suppose that A_{ij} are triangular matrices. In this case the matrix

$$(q - q^{-1})^{-1}(A_{11}A_{22} - A_{22}A_{11}) \quad (39)$$

has only zero entries on the main diagonal. This matrix is equal to $A_{12}A_{21}$. From this follows that the main diagonal of $A_{11}A_{22}$ and that of the invertible matrix $\det_q = A_{11}A_{12} - qA_{12}A_{21}$ coincide. This means that A_{11} and A_{22} have no zero terms on the main diagonal and therefore they are invertible. \square

Now we are ready to answer the question when inner actions defined by two different representations are equivalent to each other.

Theorem 2 . Let $a_{ij} \rightarrow A_{ij}$ and $a_{ij} \rightarrow A'_{ij}$ be two representations of $GL_q(2, C)$ in $C(1, 3)$. Then Hopf algebra actions

$$a_{ij} \cdot v = \sum_k A_{ik} v A_{kj}^* \quad (40)$$

and

$$a_{ij} * v = \sum_k A'_{ik} v A_{kj}'^* \quad (41)$$

are equivalent if and only if

$$\begin{aligned} A'_{11} &= u A_{11} u^{-1} \alpha_1, \quad A'_{12} = u A_{12} u^{-1} \alpha_2, \\ A'_{21} &= u A_{21} u^{-1} \alpha_1, \quad A'_{22} = u A_{22} u^{-1} \alpha_2, \end{aligned} \quad (42)$$

for some nonzero complex numbers α_1, α_2 and invertible $u \in C(1, 3)$. Formulas (f96) and (f97) define the same action if and only if the elements A'_{ij} are connected with A_{ij} by formulas (f98), with $u = 1$.

Proof. First of all, it is easy to see that if $a_{ij} \rightarrow A_{ij}$ is a representation and A'_{ij} is defined by (98), then $a_{ij} \rightarrow A'_{ij}$ is also a representation of $GL_q(2, C)$. In order to show that they define equivalent actions let us present formulas (f96), (f97) in matrix form

$$\begin{pmatrix} a_{11} \cdot v & a_{12} \cdot v \\ a_{21} \cdot v & a_{22} \cdot v \end{pmatrix} = A \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} A^{-1} \quad (43)$$

and

$$\begin{pmatrix} a_{11} * v & a_{12} * v \\ a_{21} * v & a_{22} * v \end{pmatrix} = A_1 \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} A_1^{-1}, \quad (44)$$

where by Theorem 1

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} A_{11}^* & A_{12}^* \\ A_{21}^* & A_{22}^* \end{pmatrix} \quad (45)$$

and

$$A_1 = \begin{pmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{pmatrix} = \begin{pmatrix} u A_{11} u^{-1} \alpha_1 & u A_{12} u^{-1} \alpha_2 \\ u A_{21} u^{-1} \alpha_1 & u A_{22} u^{-1} \alpha_2 \end{pmatrix} = \quad (46)$$

$$\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} A \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}.$$

From this formula we have

$$A_1^{-1} = \begin{pmatrix} \alpha_1^{-1} & 0 \\ 0 & \alpha_2^{-1} \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} A^{-1} \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix}. \quad (47)$$

Taking into account that $\text{diag}(v, v)$ commute with all 2×2 matrix with complex coefficients (and in particular with $\text{diag}(\alpha_1, \alpha_2)$) we have

$$\begin{aligned} A_1 \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} A_1^{-1} &= \\ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} A \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} A^{-1} \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} &= \\ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} A \begin{pmatrix} u^{-1}vu & 0 \\ 0 & u^{-1}vu \end{pmatrix} A^{-1} \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} &= \\ \begin{pmatrix} ua_{11} \cdot (u^{-1}vu)u^{-1} & ua_{12} \cdot (u^{-1}vu)u^{-1} \\ ua_{21} \cdot (u^{-1}vu)u^{-1} & ua_{22} \cdot (u^{-1}vu)u^{-1} \end{pmatrix}. \end{aligned} \quad (48)$$

If we set $w = u^{-1}vu$, then by(f100) and(f104) we will get

$$a_{ij} * (uwu^{-1}) = u(a_{ij} \cdot w)u^{-1}. \quad (49)$$

The last formula concides with(fequiv) and therefore the actions are equivalent. If $u = 1$, then(f105) shows that the actions coincide.

Inversely, let $a_{ij} \rightarrow A_{ij}$, $a_{ij} \rightarrow A'_{ij}$ be two representations which define equivalent (or equal) actions. Then we have the relations of type(fequiv) (with $u = 1$, if the actions coincide). In the matrix from for $v = uwu^{-1}$ this relations are equivalent to

$$\begin{aligned} A_1 \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} A_1^{-1} &= \\ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} A \begin{pmatrix} u^{-1}vu & 0 \\ 0 & u^{-1}vu \end{pmatrix} A^{-1} \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} \end{aligned} \quad (50)$$

Let us multiply this relation from the left by

$$\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} A^{-1} \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix}$$

and from the right by A_1 . We will get

$$\begin{aligned} & \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} A^{-1} \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} A_1 \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} = \\ & \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} A^{-1} \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} A_1. \end{aligned} \quad (51)$$

If we denote by α_{ij} the coefficients of $\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} A^{-1} \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} A_1$, then this relation reduces to

$$\begin{pmatrix} \alpha_{11}v & \alpha_{12}v \\ \alpha_{21}v & \alpha_{22}v \end{pmatrix} = \begin{pmatrix} v\alpha_{11} & v\alpha_{12} \\ v\alpha_{21} & v\alpha_{22} \end{pmatrix}, \quad (52)$$

i.e. $[\alpha_{ij}, v] = 0$. As v is an arbitrary element from $C(1, 3)$, the elements α_{ij} belong to the center of $C(1, 3)$. The center of $C(1, 3)$ coincide with the set of all complex numbers, i.e. $\alpha_{ij} \in C$. Thus, we have

$$A_1 = \begin{pmatrix} u^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} A \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \quad (53)$$

or in details

$$A'_{11} = u^{-1}(A_{11}\alpha_{11} + A_{12}\alpha_{21})u, \quad A'_{22} = u^{-1}(A_{21}\alpha_{12} + A_{22}\alpha_{22})u \quad (54)$$

$$A'_{12} = u^{-1}(A_{11}\alpha_{12} + A_{12}\alpha_{22})u, \quad A'_{21} = u^{-1}(A_{21}\alpha_{11} + A_{22}\alpha_{21})u. \quad (55)$$

We know that $a_{ij} \rightarrow A'_{ij}$ is a representation of $GL_q(2, C)$ and therefore by Corollary 3 the elements A'_{12} , A'_{21} are nilpotent. By Corollary 2 we can suppose that A_{ij} are triangular matrices of the form (90) and by Corollary 3 all entries on the main diagonal of A_{11} , A_{22} are nonzero, while all entries on the main diagonal of A_{12} , A_{21} are zero. Now, if $\alpha_{12} \neq 0$ then the first

formula of(f111) shows that all entries on the main diagonal of $uA'_{12}u^{-1}$ are nonzero, therefore $uA'_{12}u^{-1}$ as well as A'_{12} are invertible. This is impossible as A'_{12} is nilpotent. So $\alpha_{12} = 0$.

In the same way $\alpha_{21} = 0$. Thus, if we denote $\alpha_1 = \alpha_{11}$, $\alpha_2 = \alpha_{22}$ then(f110) and(f111) reduce to(f98). \square

Definition. We call two representations $a_{ij} \rightarrow A_{ij}$, $a_{ij} \rightarrow A'_{ij}$ equivalent if they define equivalent actions of $GL_q(2, C)$; i.e., if relations (f98) are satisfied.

As a homomorphism $\varphi : GL_q(2, C) \rightarrow C(1, 3)$ defines a four dimensional left module over the algebra $GL_q(2, C)$ and viceversa, Theorem t4 shows that the equivalence of representations means that corresponding modules V_1 , V_2 are related by formula $V_1 \simeq V_2 \otimes U$, where U is any one dimensional module.

Indeed, numbers α_1 , α_2 define one dimensional representation $a_{11} \rightarrow \alpha_1$, $a_{12} \rightarrow 0$, $a_{21} \rightarrow 0$, $a_{22} \rightarrow \alpha_2$ and every one dimensional representation $a_{ij} \rightarrow \alpha_{ij}$ is defined by two nonzero numbers α_{11} , α_{22} , while $\alpha_{12}=\alpha_{21}=0$ ($\alpha_{11}\alpha_{12}=q\alpha_{12}\alpha_{11}=q\alpha_{11}\alpha_{12}$, $\alpha_{11} \neq 0$).

5 Algebra R_q and its representations

Let us consider some simplification of the algebraic structure of $GL_q(2, C)$. Let the algebra R_q be generated by the elements $a_{11}, a_{12}, a_{21}, r_{22}$ with the relations between a_{11}, a_{12}, a_{21} defined by (p1) and the additional relations

$$a_{12}r_{22} = qr_{22}a_{12}, \quad a_{21}r_{22} = qr_{22}a_{21}, \quad (56)$$

$$a_{11}r_{22} = r_{22}a_{11}. \quad (57)$$

In fact this means that R_q is defined by the same system of spinors with commutative diagonal:

$$\begin{array}{ccc}
 a_{11} & \longrightarrow & a_{12} \\
 \downarrow & \searrow & \downarrow \\
 a_{21} & \longrightarrow & r_{22}
 \end{array} \tag{58}$$

Theorem 3 . If $a_{ij} \rightarrow A_{ij}$ is a finite dimensional representation of the algebra $GL_q(2, C)$, $q^m \neq 1$, then

$$a_{ij} \rightarrow A_{ij}, \quad r_{22} \rightarrow R_{22} = A_{22} - A_{12}A_{11}^{-1}A_{21} \tag{59}$$

is a representation of R_q with invertible $A_{11}R_{22}$. Inversely, if $a_{ij} \rightarrow A_{ij}$ ($i \neq 2$ or $j \neq 2$), $r_{22} \rightarrow R_{22}$ is a finite dimensional representation of R_q , such that A_{11} , R_{22} are invertible then

$$a_{ij} \rightarrow A_{ij}, \quad a_{22} \rightarrow A_{22} = R_{22} + A_{12}A_{11}^{-1}A_{21} \tag{60}$$

is a representation of algebra $GL_q(2, C)$. In this case $(rep_{gl}q)$ is a representation of $SL_q(2, C)$ iff $R_{22} = A_{11}^{-1}$.

Proof. Let $a_{ij} \rightarrow A_{ij}$ be a finite dimensional representation of $GL_q(2, C)$. Then by Corollary c3 the elements A_{11} , A_{22} are invertible, therefore formula (repspi) is correctly defined. It is easy to see that

$$[A_{12}, R_{22}]_q = [A_{21}, R_{22}]_q = [A_{11}, R_{22}] = 0, \tag{61}$$

where we have denoted $[x, y]_q = xy - qyx$, $[x, y] = xy - yx$.

Besides we have

$$R_{22} = A_{22} - qA_{11}^{-1}A_{12}A_{21} = A_{11}^{-1}(A_{11}A_{22} - qA_{12}A_{21}) = A_{11}^{-1}det_q,$$

which shows that R_{22} is invertible.

Inversely, let $r_{22} \rightarrow R_{22}$, $a_{ij} \rightarrow A_{ij}$; $i \neq 2$ or $j \neq 2$, be a finite dimensional representation of R_q with invertible A_{11} , R_{22} . Straightforward calculations show that

$$[A_{12}, A_{22}]_q = [A_{21}, A_{22}]_q = 0, \quad [A_{11}, A_{22}] = (q - q^{-1})A_{12}A_{21}. \quad (62)$$

We have

$$\begin{aligned} \det_q &= A_{11}A_{22} - qA_{12}A_{21} = A_{11}(R_{22} + A_{12}A_{11}^{-1}A_{21}) - qA_{12}A_{21} = \\ &= A_{11}R_{22} + qA_{12}A_{21} - qA_{12}A_{21} = A_{11}R_{22}, \end{aligned}$$

therefore \det_q is invertible. \square

6 Actions defined by $SL_q(2, C)$ -representations

For a classification of the algebra representations of $GL_q(2, C)$ in $C(1, 3)$, by Theorem 1, it is enough to describe representations of R_q with invertible a_{11} , r_{22} . In this case we have two q -spinors (a_{11}, a_{12}) , (a_{11}, a_{21}) and two q^{-1} -spinors (r_{22}, a_{12}) , (a_{22}, a_{21}) with invertible first terms, such that $a_{11}r_{22} = r_{22}a_{11}$.

Note that if (x, y) is a q -spinor then (x^{-1}, y) is a q^{-1} -spinor. Therefore for each of the two representations $(x, y) \rightarrow (A_{11}, A_{12})$, $(x, y) \rightarrow (A_{11}, A_{21})$ with invertible A_{11} and commuting A_{12} , A_{21} we have a representation of the algebra R_q :

$$a_{11} \rightarrow A_{11}, \quad a_{12} \rightarrow A_{12}, \quad a_{21} \rightarrow A_{21}, \quad r_{22} \rightarrow A_{11}^{-1}. \quad (63)$$

By Theorem spin this representation defines a representation of $SL_q(2, C)$. In this way, using q -spinor representations given in Theorem t2, we can write a number of $SL_q(2, C)$ -representations with nontrivial “perturbation” — see all representations in the Table, which are marked by the letter S : $S1, S2a, S2a'$, e.t.c.

Theorem 4 . *Every representation $a_{ij} \rightarrow A_{ij}$ of the algebra $SL_q(2, C)$, $q^m \neq 1$ in $C(1, 3)$ with nontrivial “perturbation” is equivalent to one of the representations marked by letter S in the Table. Invariants and operator algebras of corresponding inner actions are given in the Table.*

Proof. Let $a_{ij} \rightarrow A_{ij}$ be a representation of $GL_q(2, C)$ and $a_{ij} \rightarrow A_{ij}$, $r_{22} \rightarrow R_{22}$ be the corresponding representation of R_q ; i.e. $R_{22}=A_{22} - qA_{11}^{-1}A_{12}A_{21}$ (see (repspi)). By Theorem 1 we have $R_{22}=A_{11}^{-1}$ and therefore

$$A_{22} = A_{11}^{-1}(1 + qA_{12}A_{21}). \quad (64)$$

We know that $a_{11} \rightarrow A_{11}$, $a_{21} \rightarrow A_{21}$ and $a_{11} \rightarrow A_{11}$, $a_{12} \rightarrow A_{12}$ are two representations of the q -spinor with an invertible first term. Recall that for a matrix A we have denoted by $B(A)$ the linear space of all matrices B such that $AB = qBA$. Thus $A_{12}, A_{21} \in B(A_{11})$ and so $B(A_{11})^2 \neq 0$. By Theorem t2 we can assume that A_{11} is one of the seven matrices which appear in this theorem. Let us consider the seven cases separately.

1. $A_{11} = \text{diag}(q^2, q, 1, 1)$. In this case we have

$$A_{12} = \alpha e_{12} + \beta e_{23} + \gamma e_{24}, \quad A_{21} = \alpha_1 e_{12} + \beta_1 e_{23} + \gamma_1 e_{24}, \quad (65)$$

therefore

$$A_{12}A_{21} = \alpha\beta_1 e_{13} + \alpha\gamma_1 e_{14} = A_{21}A_{12} = \alpha_1\beta e_{13} + \alpha_1\gamma e_{14} \neq 0. \quad (66)$$

These relations imply that

$$0 = \alpha\beta_1 - \alpha_1\beta = \det \begin{pmatrix} \alpha & \beta \\ \alpha_1 & \beta_1 \end{pmatrix}, \quad 0 = \alpha\gamma_1 - \alpha_1\gamma = \det \begin{pmatrix} \alpha & \gamma \\ \alpha_1 & \gamma_1 \end{pmatrix}. \quad (67)$$

The first of equalities (s1.3) shows that (α, β) and (α_1, β_1) are linearly dependent vectors and by (s1.2) we can write $(\alpha_1, \beta_1) = \frac{\alpha_1}{\alpha}(\alpha, \beta)$. In the same way $(\alpha_1, \gamma_1) = \frac{\alpha_1}{\alpha}(\alpha, \gamma)$. These two relations are equivalent to

$$A_{21} = \epsilon A_{12}, \quad \epsilon = \frac{\alpha_1}{\alpha} \neq 0. \quad (68)$$

Let us consider a matrix of the form $U = \text{diag}(1, d, M)$, where M is an invertible 2×2 matrix and d a nonzero complex number. This matrix commutes with A_{11} therefore the conjugation by it will not change A_{11} , while

A_{12} is changed in the following way

$$UA_{12}U^{-1} = \begin{pmatrix} 0 & \alpha d^{-1} & 0 & 0 \\ 0 & 0 & (d\beta, \alpha\gamma)M^{-1} & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (69)$$

Let us take $d = \alpha$. Then $(d\beta, \alpha\gamma) = (\alpha\beta, \alpha\gamma) \neq 0$ and there exists an invertible 2×2 matrix M such that $(\alpha\beta, \alpha\gamma)M^{-1} = (1, 0)$. If we replace ϵ by α , then we will get the representation $S1$ of the Table.

Let us calculate \mathfrak{R} and the invariants of the $SL_q(2, C)$ -action defined by this representation.

All three degrees of A_{11} are linearly independent, as the Vandermond determinant does not vanish. This means that the algebra \mathfrak{R} contains the elements $e_{11}, e_{22}, e_{33} + e_{44}$. We have $A_{12} = e_{12} + e_{23} \in \mathfrak{R}$ and $e_{11}A_{12} = e_{12} \in \mathfrak{R}$. Thus $e_{12}, e_{23}, e_{13} = e_{12}e_{23} \in \mathfrak{R}$. The linear space generated by these six elements is a subalgebra given in S1 in the Table, which evidently is isomorphic to the algebra of triangular 3×3 matrices T_3 . For calculating the invariants we can use Lemma 1. It is easy to see that the centralizer of \mathfrak{R} is equal to the algebra of matrices $\beta E + \gamma e_{44} + \delta e_{43}$, which is isomorphic to the algebra of 2×2 triangular matrices. In fact in this case we have just two basic nontrivial invariants (see matrix)

$$I_1 = (1 - \gamma_0)(-\gamma_1 + i\gamma_2)\gamma_3, \quad I_2 = (1 - \gamma_0)(1 - i\gamma_{12}), \quad (70)$$

while all others are linear combinations of them and of the unit element.

2. $A_{11} = \text{diag}(q^2, q, q, 1)$. By theorem t2 we have $A_{12} = \alpha e_{12} + \beta e_{13} + \gamma e_{24} + \delta e_{34}$, $A_{21} = \alpha_1 e_{12} + \beta_1 e_{13} + \gamma_1 e_{24} + \delta_1 e_{34}$. Therefore

$$A_{12}A_{21} = (\alpha\gamma_1 + \beta\delta_1)e_{14} = A_{21}A_{12} = (\alpha_1\gamma + \beta_1\delta)e_{14} \neq 0. \quad (71)$$

From (s2.1) we have

$$\alpha\gamma_1 + \beta\delta_1 = \alpha_1\gamma + \beta_1\delta \neq 0. \quad (72)$$

The matrix A_{11} commutes with all matrices $U = \text{diag}(d, M, 1)$, where d is a nonzero complex number and M is an invertible 2×2 matrix. Conjugation by a matrix U does not change A_{11} , while A_{12} , A_{21} are changed by the following formulae

$$\begin{aligned} A_{12} &\rightarrow \begin{pmatrix} 0 & (d\alpha, d\beta)M^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} M \begin{pmatrix} \gamma \\ \delta \end{pmatrix}; \\ A_{21} &\rightarrow \begin{pmatrix} 0 & (d\alpha_1, d\beta_1)M^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} M \begin{pmatrix} \gamma_1 \\ \delta_1 \end{pmatrix}. \end{aligned} \quad (73)$$

In particular, the matrix $\begin{pmatrix} \gamma & \gamma_1 \\ \delta & \delta_1 \end{pmatrix}$ under this conjugation is multiplied by M from the left. For this matrix we have two possibilities.

a) $\det \begin{pmatrix} \gamma & \gamma_1 \\ \delta & \delta_1 \end{pmatrix} \neq 0$. In this case this matrix is invertible and we can take M to be its inverse and (Suemi) reduces to the form $A_{12} \rightarrow d\alpha e_{12} + d\beta e_{13} + e_{24}$, $A_{21} \rightarrow d\alpha_1 e_{12} + d\beta_1 e_{13} + e_{34}$, with new parameters $\alpha, \beta, \alpha_1, \beta_1$. Relations (s2.2) become $\alpha \cdot 0 + \beta \cdot 1 \neq 0$; i.e. $\beta \neq 0$, and $\beta = \alpha_1 \cdot 1 + \beta_1 \cdot 0 = \alpha_1 \neq 0$. If we let $d = \beta^{-1} = \alpha_1^{-1}$, then changing $\alpha_1^{-1}\alpha \rightarrow \alpha$, $\alpha_1^{-1}\beta_1 \rightarrow \beta$, we get the representation $S2a$.

For calculating the algebra \mathfrak{R} we can make the same procedure as in the first case. The first three degrees of A_{11} are linearly independent and so \mathfrak{R} contains $e_{11}, e_{22} + e_{33}, e_{44}$. Multiplication of A_{12} and A_{21} by e_{44} from the right gives two elements e_{24}, e_{34} and also $\alpha e_{12} + e_{13}, e_{12} + \beta e_{13}$.

If $\alpha\beta \neq 1$ then these two elements are linearly independent and $e_{12}, e_{13} \in \mathfrak{R}$. So \mathfrak{R} consists of matrices of the form given in the Table for representation $S2a$.

If $\alpha\beta = 1$ then the elements $\alpha e_{12} + e_{13}$ and $e_{12} + \beta e_{13}$ are linearly dependent and all matrices from \mathfrak{R} have $(-_{12}, -_{13})$ -entries proportional to $(\alpha, 1) = \alpha(1, \beta)$; i.e. \mathfrak{R} has the form given in the Table for $S2a'$. Easy calculations show that in both cases the centralizer of \mathfrak{R} is equal to $1 \cdot C$.

b) $\det \begin{pmatrix} \gamma & \gamma_1 \\ \delta & \delta_1 \end{pmatrix} = 0$. By the relation (s2.1) we have $\begin{pmatrix} \gamma \\ \delta \end{pmatrix} \neq 0$, $\begin{pmatrix} \gamma_1 \\ \delta_1 \end{pmatrix} \neq 0$, therefore it is possible to find a matrix M_1 such that

$$M_1 \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad M_1 \begin{pmatrix} \gamma_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} \gamma'_1 \\ 0 \end{pmatrix}, \quad \gamma'_1 \neq 0. \quad (74)$$

Let $M = M_2 M_1$, where M_2 is an invertible matrix, such that $M_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; i.e. $M_2 = \begin{pmatrix} 1 & u' \\ 0 & v \end{pmatrix}$, $v \neq 0$. In this case formula (Suemi) takes the form

$$\begin{aligned} A_{12} &\rightarrow \begin{pmatrix} 0 & (d\alpha', d\beta')M_2^{-1} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \\ A_{21} &\rightarrow \begin{pmatrix} 0 & (d\alpha'_1, d\beta'_1)M_2^{-1} & 0 \\ 0 & 0 & 0 & \gamma'_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (75)$$

where $(\alpha', \beta') = (\alpha, \beta)M_1^{-1}$, $(\alpha'_1, \beta'_1) = (\alpha_1, \beta_1)M_1^{-1}$ and

$$M_2^{-1} = \begin{pmatrix} 1 & u \\ 0 & v^{-1} \end{pmatrix}, \quad u = -u'v^{-1}. \quad (76)$$

Shortly we can write (sue) in the form

$$A_{12} \rightarrow d\alpha''e_{12} + d\beta''e_{13} + e_{24}, \quad A_{21} \rightarrow d\alpha''_1e_{12} + d\beta''_1e_{13} + \gamma''_1e_{24}, \quad (77)$$

where

$$(\alpha'', \beta'') = (\alpha', \beta') \begin{pmatrix} 1 & u \\ 0 & v^{-1} \end{pmatrix} = (\alpha', \alpha'u + \beta'v^{-1}) \quad (78)$$

and

$$(\alpha''_1, \beta''_1) = (\alpha'_1, \beta'_1) \begin{pmatrix} 1 & u \\ 0 & v^{-1} \end{pmatrix} = (\alpha'_1, \alpha'_1u + \beta'_1v^{-1}). \quad (79)$$

By using (s2.2), applied to primed parameters, we have

$$\alpha' \gamma'_1 + \beta' \cdot 0 = \alpha'_1 \cdot 1 + \beta'_1 \cdot 0 \neq 0; \text{ i.e. } \alpha' \neq 0, \quad \alpha' \gamma'_1 = \alpha'_1. \quad (80)$$

If we suppose $u = -(\alpha')^{-1} \beta' v^{-1}$ then we get $\beta'' = 0$ and

$$\beta''_1 = -\alpha'_1 (\alpha')^{-1} \beta' v^{-1} + \beta'_1 v^{-1} = (\beta'_1 - \gamma'_1 \beta') v^{-1}. \quad (81)$$

If $\beta'_1 \neq \gamma'_1 \beta'$ in formula (last), then we can take $v = (\beta'_1 - \gamma'_1 \beta') (\alpha')^{-1}$. In this case $\beta''_1 = \alpha'$ and we can let $d = (\alpha')^{-1}$ in (suem) in order to obtain $S2b$ (where by α we mean $(\alpha')^{-1} \alpha'_1 = (\alpha')^{-1} \alpha'_1 = \gamma'_1$).

If $\beta'_1 = \gamma'_1 \beta'$ in (last), then $\beta''_1 = 0$ and (suem) with $d = (\alpha')^{-1}$ is equal to $S2b'$. In this case the algebra \mathfrak{R} contains the elements $e_{11}, e_{22} + e_{33}, e_{44}, e_{12}, e_{24}, e_{14}$ which form a basis of this algebra. This algebra is isomorphic to T_3 (see in the Table an action $S2b'$.)

The centralizer of \mathfrak{R} is equal to the set of diagonal matrices of the form $\text{diag}(\beta, \beta, \delta, \beta)$; therefore it is isomorphic to the direct sum $C \oplus C$. It is interesting to note that in this case the algebra \mathfrak{R} is abstractly isomorphic to the algebra \mathfrak{R} for representation $S1$, but they are not conjugate in $C(1, 3)$ because they have nonisomorphic centralizers.

Thus, we have that the action of the quantum group $SL_q(2, C)$ defined by the representation $S2b'$ has only one basic invariant; i.e.

$$I = (1 - \gamma_0)(1 + i\gamma_{12}). \quad (82)$$

For the representation $S2b$ the algebra \mathfrak{R} contains one more matrix, e_{13} , therefore it is a 7-dimensional algebra (see the Table), whose centralizer is equal to $C \cdot 1$ and the corresponding $SL_q(2, C)$ -action has only trivial invariants.

3. $A_{11} = \text{diag}(\mathbf{q}^2, \mathbf{q}^2, \mathbf{q}, 1)$. By theorem t2 we have

$$A_{12} = \alpha e_{13} + \beta e_{23} + \gamma e_{34}, \quad A_{21} = \alpha_1 e_{13} + \beta_1 e_{23} + \gamma_1 e_{34}; \quad (83)$$

thus

$$A_{12} A_{21} = \alpha \gamma_1 e_{14} + \beta \gamma_1 e_{24} = A_{21} A_{12} = \alpha_1 \gamma e_{14} + \beta_1 \gamma e_{24} \neq 0. \quad (84)$$

These relations imply

$$0 = \alpha\gamma_1 - \alpha_1\gamma = \det \begin{pmatrix} \alpha & \gamma \\ \alpha_1 & \gamma_1 \end{pmatrix}, \quad 0 = \beta\gamma_1 - \gamma\beta_1 = \det \begin{pmatrix} \beta & \gamma \\ \beta_1 & \gamma_1 \end{pmatrix}, \quad (85)$$

$$(\alpha, \beta) \neq 0, \quad \gamma_1 \neq 0, \quad (\alpha_1, \beta_1) \neq 0, \quad \gamma \neq 0. \quad (86)$$

From these relations we have

$$A_{21} = \epsilon A_{12}, \quad \epsilon = \gamma_1/\gamma \neq 0. \quad (87)$$

Let us consider a matrix of the form $U = \text{diag}(M, d, 1)$, where M is an invertible 2×2 matrix and d is a nonzero complex number. The conjugation by this matrix does not change A_{11} but A_{12} changes in the following way

$$UA_{12}U^{-1} = \begin{pmatrix} 0 & 0 & M \begin{pmatrix} \alpha d^{-1} \\ \beta d^{-1} \end{pmatrix} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d\gamma \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (88)$$

If we take $d = \gamma^{-1}$ then $(\alpha d^{-1}, \beta d^{-1}) = (\alpha\gamma, \beta\gamma)$ is a nonzero vector. Thus there exists an invertible matrix M such that $M \begin{pmatrix} \alpha\gamma \\ \beta\gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In this way we have obtained the representation $S3$, where by α is denoted the parameter ϵ , see (s3.4).

In this case the algebra \mathfrak{R} is generated by the elements $e_{11} + e_{22}$, e_{33} , e_{44} , e_{13} , e_{34} , e_{14} ; i.e. it consists of matrices

$$\begin{pmatrix} \epsilon & 0 & * & * \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}. \quad (89)$$

This algebra is isomorphic to T_3 , while its centralizer is the algebra of matrices of the form $\beta E + \gamma e_{22} + \delta e_{12}$ this is isomorphic to the algebra of triangular 2×2 matrices T_2 . Now formulae (matrix) show that the action defined by this representation has the following basic invariants

$$I_1 = (1 + \gamma_1)(1 - i\gamma_{12}), \quad I_2 = (1 + \gamma_0)(\gamma_1 + i\gamma_2)\gamma_3. \quad (90)$$

4. $\mathbf{A}_{11} = \text{diag}(\mathbf{q}^3, \mathbf{q}^2, \mathbf{q}, 1)$. By theorem t2 we have

$$A_{12} = \alpha e_{12} + \beta e_{23} + \gamma e_{34}, \quad A_{21} = \alpha_1 e_{12} + \beta_1 e_{23} + \gamma_1 e_{34}; \quad (91)$$

thus

$$A_{12}A_{21} = \alpha\beta_1 e_{13} + \beta\gamma_1 e_{24} = A_{21}A_{12} = \alpha_1\beta e_{13} + \beta_1\gamma e_{24} \neq 0. \quad (92)$$

These relations imply

$$0 = \alpha\beta_1 - \alpha_1\beta = \det \begin{pmatrix} \alpha & \beta \\ \alpha_1 & \beta_1 \end{pmatrix}, \quad 0 = \beta\gamma_1 - \beta_1\gamma = \det \begin{pmatrix} \beta & \gamma \\ \beta_1 & \gamma_1 \end{pmatrix}. \quad (93)$$

Therefore the vectors (α, β) and (α_1, β_1) are linearly dependent and so are (β, γ) and (β_1, γ_1) . By (s4.1), one of the numbers β, β_1 is nonzero. If, for example, $\beta \neq 0$ then $(\alpha_1, \beta_1) = \frac{\beta_1}{\beta}(\alpha, \beta)$ and $(\beta_1, \gamma_1) = \frac{\beta_1}{\beta}(\beta, \gamma)$ so

$$A_{21} = \epsilon A_{12}, \quad \epsilon = \frac{\beta_1}{\beta}, \quad (94)$$

where $\epsilon \neq 0$ as $A_{21} \neq 0$. In the same way, if $\beta_1 \neq 0$, then we get the following relation

$$A_{12} = \epsilon' A_{21}, \quad \epsilon' = \frac{\beta}{\beta_1} \neq 0. \quad (95)$$

Thus, in both cases we have (s4.3) with $\epsilon \neq 0$ and $\beta, \beta_1 \neq 0$.

If $\alpha, \gamma \neq 0$ then the conjugation by a diagonal matrix $\text{diag}(1, \alpha, \alpha\beta, \alpha\beta\gamma)$ does not change A_{11} , while A_{12} is reduced to $e_{12} + e_{23} + e_{34}$; i.e. we obtain S4a. In this case the algebra \mathfrak{R} contains the elements $e_{11}, e_{22}, e_{33}, e_{44}$ (as first four powers of A_{11} are linearly independent) and $e_{12}=e_{11}A_{12}, e_{23}=e_{22}A_{12}, e_{34}=e_{33}A_{12}$ as well as elements $e_{13}=e_{12}e_{23}, e_{14}=e_{13}e_{34}, e_{24}=e_{23}e_{34}$. This means that \mathfrak{R} contains all triangular 4×4 matrices and it has the maximal possible dimension.

If $\alpha \neq 0, \gamma=0$ then, the conjugation by a diagonal matrix $\text{diag}(1, \alpha, \alpha\beta, 1)$, gives us the representation S4b. The algebra \mathfrak{R} is generated by the elements $e_{11}, e_{22}, e_{33}, e_{44}, e_{12}, e_{23}, e_{13}$. So this is a 7-dimensional algebra of matrices

(see the Table) which is isomorphic to a direct sum $T_3 \oplus C$. The centralizer consists of diagonal matrices $diag(\beta, \beta, \beta, \delta)$ and is isomorphic to $C \oplus C$. Formulas (matrix) show that the action defined by this representation have only one basic invariant; i.e.

$$I = (1 - \gamma_0)(1 - i\gamma_{12}). \quad (96)$$

If $\gamma \neq 0$, $\alpha = 0$, then the conjugation by a diagonal matrix $diag(1, 1, \beta, \beta\gamma)$ gives us the representation $S5$, where the parameter α equals q^3 (of course the parameter α in the Table is not the same as in (s4.1) and (s4.2), which is now zero).

The algebra \mathfrak{R} is generated by $e_{11}, e_{22}, e_{33}, e_{44}, e_{34}, e_{23}, e_{24}$. This is also a 7-dimensional algebra (see the Table) which is isomorphic to $T_3 \oplus C$, with the centralizer $\{diag(\gamma, \delta, \delta, \delta)\}$ isomorphic to $C \oplus C$. By formulae (matrix) we have that the action defined by this representation, has only one basic invariant; i.e.

$$I = (1 + \gamma_0)(1 + i\gamma_{12}). \quad (97)$$

5. $A_{11} = diag(\alpha, q^2, q, 1)$, $\alpha \neq 0, q^{-1}, 1, q, q^2, q^3$. In this case by theorem t2 we have

$$A_{12} = \beta e_{23} + \gamma e_{34}, \quad A_{21} = \beta_1 e_{23} + \gamma_1 e_{34} \quad (98)$$

and

$$A_{12}A_{21} = \beta\gamma_1 e_{24} = A_{21}A_{12} = \beta_1\gamma e_{24} \neq 0. \quad (99)$$

So

$$0 = \beta\gamma_1 - \beta_1\gamma = det \begin{pmatrix} \beta & \gamma \\ \beta_1 & \gamma_1 \end{pmatrix}, \quad \beta, \gamma, \beta_1, \gamma_1 \neq 0. \quad (100)$$

Therefore

$$A_{21} = \epsilon A_{12}, \quad \epsilon = \frac{\beta_1}{\beta} \neq 0. \quad (101)$$

Now the conjugation by a diagonal matrix $diag(1, 1, \beta, \beta\gamma)$ gives us the representation $S5$ and the action defined by this representation also has only one basic invariant; i.e.

$$I = (1 + \gamma_0)(1 + i\gamma_{12}). \quad (102)$$

6. $A_{11} = \text{diag}(q^2, q^2, q, 1) + e_{12}$. By theorem t2, $A_{12} = \alpha e_{13} + \beta e_{34}$, $A_{21} = \alpha_1 e_{13} + \beta_1 e_{34}$, thus

$$A_{12}A_{21} = \alpha\beta_1 e_{14} = \alpha_1\beta e_{14} \neq 0, \quad (103)$$

which implies that

$$A_{21} = \epsilon A_{12}, \quad \epsilon = \frac{\alpha_1}{\alpha} \neq 0. \quad (104)$$

Conjugation by a diagonal matrix $\text{diag}(1, 1, \alpha, \alpha\beta)$ does not change the matrix A_{11} while A_{12} is reduced by this to $e_{13} + e_{34}$ and we have the representation S6.

For the calculation of \mathfrak{R} let us note that

$$A_{11}^k = \text{diag}(q^{2k}, q^{2k}, q^k, 1) + 2^{k-1} q^{2k-2} e_{12}. \quad (105)$$

Evidently a subalgebra generated by A_{11} is contained in a four dimensional algebra of matrices generated by $e_{11} + e_{22}$, e_{33} , e_{44} , e_{12} . In order to show that these algebras are equal to each other it is enough to show that E , A_{11} , A_{11}^2 , A_{11}^3 are linearly independent. If this is not the case then A_{11} is a root of some polynomial $f(x)$ of degree three. This polynomial must be a divisor of the characteristic polynomial of A_{11} , which is equal to

$$(x - q^2)^2(x - q)(x - 1). \quad (106)$$

The polynomial (s6.4) has just three divisors of degree three and none of them has A_{11} as its root (if $q \neq \pm 1$).

Thus the algebra \mathfrak{R} contains the elements $e_{11} + e_{22}$, e_{33} , e_{44} , e_{12} , $e_{13} = A_{12}e_{33}$, $e_{34} = A_{12}e_{44}$, $e_{14} = e_{13}e_{34}$ and is generated by these elements like a linear space; i.e. it is the 7-dimensional algebra of matrices presented in the Table. Its centralizer is the two dimensional algebra $C + Ce_{12} \cong T'_2$ i.e. by formulae (matrix) the action defined by this representation has only one basic invariant; i.e.

$$I = (1 + \gamma_0)(\gamma_1 + i\gamma_2)\gamma_3. \quad (107)$$

7. $\mathbf{A}_{11} = \mathbf{diag}(\mathbf{q}^2, \mathbf{q}, \mathbf{1}, \mathbf{1}) + \mathbf{e}_{34}$. By theorem t2, we have $A_{12} = \alpha e_{12} + \beta e_{24}$, $A_{21} = \alpha_1 e_{12} + \beta_1 e_{24}$ and

$$A_{12}A_{21} = \alpha\beta_1 e_{14} = A_{21}A_{12} = \alpha_1\beta e_{14} \neq 0, \quad (108)$$

which immediately implies that

$$A_{21} = \epsilon A_{12}, \quad \epsilon = \frac{\alpha_1}{\alpha} \neq 0. \quad (109)$$

A conjugation by the matrix $diag(\alpha^{-1}\beta^{-1}, \beta^{-1}, 1, 1)$ gives the representation $S7$.

As in the previous case, the algebra \mathfrak{R} is generated by elements e_{11} , e_{22} , $e_{33} + e_{44}$, e_{34} , e_{12} , e_{24} , e_{14} , i.e. this is the algebra of matrices given in the Table and its centralizer is equal to $C + e_{34}C \cong T'_2$; i.e. the action defined by this representation has only one basic invariant

$$I = (1 - \gamma_0)(\gamma_1 + i\gamma_2)\gamma_3. \quad (110)$$

Thus the Theorem slq is proved. \square

7 Actions defined by $GL_q(2, C)$ -representations

Now let us consider representations of $GL_q(2, C)$ which are not equivalent to representations of $SL_q(2, C)$. If

$$a_{ij} \rightarrow A_{ij} \quad (111)$$

is such a representation then we can define a representation of $SL_q(2, C)$ setting $a_{ij} \rightarrow A_{ij}$ for $i \neq 2$ or $j \neq 2$ and (see formula (s))

$$a_{22} \rightarrow A'_{22} = A_{11}^{-1}(1 + qA_{12}A_{21}). \quad (112)$$

We will call these two representation *connected* to each other. Respectively, inner actions defined by connected representations will also be called *connected*.

If we denote by D the quantum determinant of the given $GL_q(2, C)$ -representation

$$D = A_{11}A_{22} - qA_{12}A_{21}, \quad (113)$$

then we obtain

$$A_{22} = A_{11}^{-1}(D + qA_{12}A_{21}) = A'_{22} + A_{11}^{-1}(D - 1). \quad (114)$$

In the last formula, D is an invertible matrix which commute with all A_{ij} . In particular it is an invariant for the connected $SL_q(2, C)$ -action (119).

Inversely, suppose that

$$a_{ij} \rightarrow A_{ij}, \quad i \neq 2 \text{ or } j \neq 2, \quad a_{22} \rightarrow A'_{22} \quad (115)$$

is a representation of $SL_q(2, C)$ and D is an invertible invariant for the action defined by this representation. In this case, formula (121) defines a representation of $GL_q(2, C)$ connected with (122).

Thus, every $GL_q(2, C)$ -representation is completely defined by the connected $SL_q(2, C)$ -representation (which can be taken from the Table) and by an invariant D of the inner action corresponding to this connected $SL_q(2, C)$ -representation (which also can be found in the Table).

If u is an invertible invariant for the connected action (119), then u commutes with all A_{ij} , ($i \neq 2$ or $j \neq 2$). Therefore the representation (given), where A_{22} is replaced by

$$A_{22} = A'_{22} + A_{11}^{-1}(uD u^{-1} - 1) \quad (116)$$

is equivalent to (given) (see formula (121) and Theorem t4. In the same way if we multiply A_{12} and D by a nonzero scalar α_2 then we will get an equivalent representation with (probably) another connected $SL_q(2, C)$ -representation and a new D proportional to the old one.

These considerations show that for the classification, up to the equivalence of all representations of $GL_q(2, C)$ connected with a given $SL_q(2, C)$ -representation, it is enough to take just one element in every projective class

of conjugated elements of the group of invertible invariants for a connected inner $SL_q(2, C)$ -action. In particular, if an inner $SL_q(2, C)$ -action has no non-trivial invariants (like $S2a, S2b, S4a$) then all $GL_q(2, C)$ -actions connected with this action are $SL_q(2, C)$ -actions.

By Theorem slq (see representations marked by S in the Table) for the algebra of invariants of an $SL_q(2, C)$ -action we have just three nontrivial possibilities:

$$Inv \cong \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = T_2, \quad Inv \cong C \oplus C, \quad Inv \cong \begin{pmatrix} \epsilon & * \\ 0 & \epsilon \end{pmatrix} = T'_2.$$

It is easy to see that if a triangular 2×2 matrix has different nonzero elements on the diagonal, then this matrix is conjugated in T_2 with a diagonal one. If this matrix has on its diagonal elements equal to ϵ then this matrix is conjugated in T_2 with a triangular matrix whose nonzero entries are all equal to ϵ . Therefore, in the first of these three cases we have just two types of possible values for D , which belong to different projective classes of conjugated elements:

$$D = \text{diag}(1, \beta), \quad \beta \neq 0, 1; \quad D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (117)$$

In the second and third cases the algebra Inv is commutative; therefore D can be chosen, respectively, in the forms

$$D = 1 \oplus \beta, \quad \beta \neq 0, 1 \quad (118)$$

and

$$D = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}, \quad \xi \neq 0. \quad (119)$$

Thus, using Theorem slq and formula (121) with D defined by (123), (124), (125) and making usual calculations for finding the algebras \mathfrak{R} and their centralizers one can obtain the following result.

Theorem 5 . *Every representation $a_{ij} \rightarrow A_{ij}$ of the algebra $GL_q(2, C)$, $q^m \neq 1$, in $C(1, 3)$ with nontrivial perturbation is equivalent to one of the representations given in the Table, which contains the operator algebras, the quantum determinants and the invariants of the corresponding inner actions.*

In order to show that the Table presents a complete classification it is necessary to prove that different representations in the Table define different inner actions.

Theorem 6 . *Two representations given in the Table are equivalent if and only if they are equal to each other.*

Proof. Let $a_{ij} \rightarrow A_{ij}$ and $a_{ij} \rightarrow A'_{ij}$ be equivalent representations given in the Table. By Theorem t2 we have

$$\begin{aligned} A'_{11} &= uA_{11}u^{-1}\alpha_1, \quad A'_{12} = uA_{12}u^{-1}\alpha_2, \\ A'_{21} &= uA_{21}u^{-1}\alpha_1, \quad A'_{22} = uA_{22}u^{-1}\alpha_2. \end{aligned} \tag{120}$$

The first of these equations shows that matrices $A'_{11}\alpha_1^{-1}$ and A_{11} have the same Jordan Normal Form. In particular, if one of the matrices A_{11}, A'_{11} is diagonal then so is the other one and $A'_{11}\alpha_1^{-1}$ can be obtained from A_{11} by permutation of its diagonal elements (note that all matrices A_{11} in the Table have a Jordan Normal Form). It easy to see that no one pair of *different* diagonal matrices A_{11} from the Table satisfies this property (here it is essential that $q^4, q^3 \neq 1$, and $\alpha \neq q^{-1}$ in S5). Thus, in these case $A'_{11} = A_{11}$.

If both matrices A_{11}, A'_{11} are not diagonal and $A_{11} \neq A'_{11}$ then one of this matrices appears in S6 while another in S7. Let us assume $A'_{11} = \text{diag}(q^2, q, 1, 1) + e_{34}$ and $A_{11} = \text{diag}(q^2, q^2, q, 1) + e_{12}$. The matrix A_{11} has a Jordan Normal Form while the Jordan Normal Form of $A'_{11}\alpha_1^{-1}$ is equal to

$$\text{diag}(\alpha_1^{-1}, \alpha_1^{-1}, q^2\alpha_1^{-1}, q\alpha_1^{-1}) + e_{12}. \tag{121}$$

Therefore $\alpha_1^{-1} = q^2$ and either $q^2\alpha_1^{-1} = q, q\alpha_1^{-1} = 1$ or $q^2\alpha_1^{-1} = 1, q\alpha_1^{-1} = q$. Both cases are impossible since $q^4, q^3 \neq 1$. Thus, we have proved that $A'_{11} = A_{11}$ in all cases.

Now we have $uA_{11}u^{-1}\alpha_1 = A_{11}$ or $uA_{11}\alpha_1 = A_{11}u$, i.e. $x \rightarrow A_{11}, y \rightarrow u$ is a representation of the α_1 -spinor with invertible both terms. This implies

that α_1 is a root of unity, $\alpha_1^m = 1$. If $(\beta_1, \beta_2, \beta_3, \beta_4)$ is a quadruple of eigenvalues (or, more exactly, the main diagonal of the Jordan Normal Form) of A_{11} , then the eigenvalues of $A_{11}\alpha_1$ form a quadruple $(\beta_1\alpha_1, \beta_2\alpha_1, \beta_3\alpha_1, \beta_4\alpha_1)$ and the equality $u^{-1}A_{11}u = A_{11}\alpha_1$ shows that these two quadruples coincide up to a permutation

$$(\beta_1, \beta_2, \beta_3, \beta_4) = (\beta_{\pi(1)}\alpha_1, \beta_{\pi(2)}\alpha_1, \beta_{\pi(3)}\alpha_1, \beta_{\pi(4)}\alpha_1), \quad (122)$$

where π is a permutation of four indices.

Now it is easy to see that no one of the matrices A_{11} given in the Table satisfies (128) for $\alpha_1^m = 1, \alpha_1 \neq 1$. Thus we have shown that $\alpha_1 = 1$ and u commutes with A_{11} .

Let us prove that $\alpha_2 = 1$. The following relation is valid for quantum determinants

$$D' = uDu^{-1}\alpha_1\alpha_2 = uDu^{-1}\alpha_2. \quad (123)$$

This relation shows that quadruples of eigenvalues of D' and D are connected by the following relation

$$(\beta'_1, \beta'_2, \beta'_3, \beta'_4) = (\beta_{\pi(1)}\alpha_2, \beta_{\pi(2)}\alpha_2, \beta_{\pi(3)}\alpha_2, \beta_{\pi(4)}\alpha_2). \quad (124)$$

It is easy to see that for every quantum determinant from the Table either all four eigenvalues are equal to 1 or three of them are equal to 1 while the fourth one is arbitrary (like in $G1a, G2, G3a, G4, G5$). In both cases the relations (130) are possible only if $\alpha_2 = 1$.

Assume now that no one of our two representations belongs to $S2a, S2a', S2b$.

All the representations from the Table not belonging to these groups have A_{12} with no parameters. If $A_{11} \neq \text{diag}(q^3, q^2, q, 1)$ then this fact immediately implies that $A'_{12} = A_{12}$. For $A_{11} = \text{diag}(q^3, q^2, q, 1)$ the element u commutes with this diagonal matrix, therefore it is itself a diagonal matrix. We see that no different values for A_{12} (that is $e_{12} + e_{23} + e_{34}; e_{12} + e_{23}; e_{23} + e_{34}$) are conjugated by a diagonal matrix, so again $A'_{12} = A_{12}$ and in all the cases $uA_{12}u^{-1} = A_{12}$, i.e. u commutes both with A_{11} and A_{12} . In all of the cases

under consideration $A_{21} = \alpha A_{12}$, therefore

$$A'_{21} = u A_{21} u^{-1} = \alpha u A_{12} u^{-1} = \alpha A_{12} = A_{21}. \quad (125)$$

By formula (119) this implies that connected $SL_q(2, C)$ -representations coincide and u belongs to the algebra of invariants of the connected $SL_q(2, C)$ -representation. So the relation $D' = u D u^{-1}$ implies $D' = D$ as by the choice of D (see (123), (124), (125)) different D, D' are not conjugated in the algebra of invariants of the connected $SL_q(2, C)$ -representation. Now, formula (121) shows that $A'_{22} = A_{22}$ and the representations coincide.

Finally, we have to consider representations with $A_{11} = \text{diag}(q^2, q, q, 1)$. Representations from distinct groups $S2a, S2a', S2b, S2b', G2$ have non-isomorphic algebras \mathfrak{R} and therefore they cannot be equivalent (recall that by (126) the algebras \mathfrak{R} for equivalent representations are conjugated by u). Thus, our representations belong to the same group and we need to consider the first three groups (in $S2b'$ and $G2$ matrices A_{12} have no parameters and A_{11}, A_{12} generate \mathfrak{R} for the connected $SL_q(2, C)$ -representation).

S2a, S2a'. In these cases A_{22} has no parameters, therefore $A'_{22} = A_{22}$ and the matrix u commutes with A_{11}, A_{22} . This means that $u = \text{diag}(1, M, 1)$, where M is an invertible 2×2 matrix.

If $A_{12} = \alpha e_{12} + e_{13} + e_{24}$ and $A'_{12} = \alpha' e_{12} + e_{13} + e_{24}$, then the relation $u A_{12} = A'_{12} u$ implies

$$M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (\alpha, 1) = (\alpha', 1) M. \quad (126)$$

The first of these equations shows that $M = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ and in this case $(\alpha', 1) M = (\alpha', *)$, so $\alpha' = \alpha$ by the second equation of (131).

In the same way, if $A_{21} = e_{12} + \beta e_{13} + e_{34}$ and $A'_{21} = e_{12} + \beta' e_{13} + e_{34}$, then

$$(1, \beta) M = (1, \beta'), \quad M \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (127)$$

and

$$\beta' = (1, \beta') \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1, \beta) M \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1, \beta) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \beta. \quad (128)$$

S2b. In this case A_{12} has no parameters, so $A'_{12} = A_{12}$ and u commutes with A_{11}, A_{12} . An algebra generated by A_{11}, A_{12} of $S2b$ is equal to \mathfrak{R} for $S2b'$, thus u is an invariant for $S2b'$ i.e. $u = \text{diag}(\beta, \beta, \delta, \beta)$ (see the Table). Conjugation by u of $A_{21} = \alpha e_{12} + \alpha e_{24} + e_{13}$ gives $A'_{21} = u A_{12} u^{-1} = \alpha e_{12} + \alpha e_{24} + \alpha \delta^{-1} e_{13}$, while $A'_{21} = \alpha' e_{12} + \alpha' e_{24} + e_{13}$, therefore $\alpha \delta^{-1} = 1$ and $\alpha = \alpha'$. \square

NOTE. In our proofs we do not need the fact that every finite dimensional representation of $GL_q(2, C)$ is triangular. We need this fact only for four or less dimensional representations. For this we need only restrictions $q^6, q^8 \neq 1$ and q can be a root of unity. Our classification is correct for $q^6, q^8 \neq 1$ (for example if $q = \exp(\frac{2\pi i}{5})$). It is easy to see that this classification is not valid if $q = \pm i$ or $q = \pm \exp(\frac{2\pi i}{3})$ as in this cases there exist, respectively, four and three dimensional irreducible representations of $SL_q(2, C)$ with $\dim \mathfrak{R} = 16, 9$, respectively. Nevertheless we do not know if this classification is correct for $q = \pm \exp(\frac{\pi i}{3})$ or $q^4 = -1$.

8 Table and corollaries

In the Table we have denoted by S (followed by some symbols) the representations of $SL_q(2, C)$ and by G (followed by the same symbols) representations of $GL_q(2, C)$, connected with the corresponding representation of $SL_q(2, C)$. We have shown in the Table five ingredients for every representation: the values of A_{ij} , the matrix form of the operator algebra \mathfrak{R} , its dimension, the invariants of the inner action defined by this representation and the value of the quantum determinant. If the value of some A_{ij} is not shown for a particular case then this means that it coincides with the value of the previous representation in the Table. By bold style we have denote seven representations which start groups with the same values of A_{11} .

Table

S1	$\mathbf{A}_{11} = \mathbf{diag}(\mathbf{q}^2, \mathbf{q}, \mathbf{1}, \mathbf{1})$ $A_{12} = e_{12} + e_{23}$ $A_{21} = \alpha A_{12}, \alpha \neq 0$ $A_{22} = A_{11}^{-1} + \alpha q^{-1} e_{13}$	$\mathfrak{R} = \begin{pmatrix} * & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \cong T_3$
$dim\mathfrak{R}$ 6	Invariants $\cong T_2$ $diag(\beta, \beta, \beta, \gamma) + \delta e_{43}$ $I_1 = (1 - \gamma_0)(-\gamma_1 + i\gamma_2)\gamma_3$ $I_2 = (1 - \gamma_0)(1 - i\gamma_{12})$	$det_q = 1$
G1a	$A_{22} = A_{11}^{-1} + \alpha q^{-1} e_{13} + \beta e_{44}$ $\alpha, \beta \neq 0, \beta \neq -1$	$\mathfrak{R} = \begin{pmatrix} * & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \cong T_3 \oplus C$
$dim\mathfrak{R}$ 7	Invariants $\cong C \oplus C$ $diag(\gamma, \gamma, \gamma, \delta)$ $I = (1 - \gamma_0)(1 - i\gamma_{12})$	$det_q = 1 + \beta e_{44}$
G1b	$A_{22} = A_{11}^{-1} + \alpha q^{-1} e_{13} + e_{43}$ $\alpha \neq 0$	$\mathfrak{R} = \begin{pmatrix} * & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & * & \epsilon \end{pmatrix}$
$dim\mathfrak{R}$ 7	Invariants $\cong T'_2$ $diag(\beta, \beta, \beta, \beta) + \delta e_{43}$ $I = (1 - \gamma_0)(-\gamma_1 + i\gamma_2)\gamma_3$	$det_q = 1 + e_{43}$

S2a	$\mathbf{A}_{11} = \mathbf{diag}(\mathbf{q}^2, \mathbf{q}, \mathbf{q}, 1)$ $A_{12} = \alpha e_{12} + e_{13} + e_{24}$ $A_{21} = e_{12} + \beta e_{13} + e_{34}$ $A_{22} = A_{11}^{-1} + q^{-1}e_{14}$ $\alpha\beta \neq 1$	$\mathfrak{R} = \begin{pmatrix} * & * & * & * \\ 0 & \epsilon & 0 & * \\ 0 & 0 & \epsilon & * \\ 0 & 0 & 0 & * \end{pmatrix}$
$\dim \mathfrak{R}$ 8	Invariants $\cong C$	$\det_q = 1$
S2a'	$\alpha\beta = 1$	$\mathfrak{R} = \begin{pmatrix} * & \alpha\gamma & \gamma & * \\ 0 & \epsilon & 0 & * \\ 0 & 0 & \epsilon & * \\ 0 & 0 & 0 & * \end{pmatrix}$
$\dim \mathfrak{R}$ 7	Invariants $\cong C$	$\det_q = 1$
S2b	$A_{11} = \mathbf{diag}(q^2, q, q, 1)$ $A_{12} = e_{12} + e_{24}$ $A_{21} = \alpha e_{12} + \alpha e_{24} + e_{13}$ $A_{22} = A_{11}^{-1} + q^{-1}\alpha e_{14}$ $\alpha \neq 0$	$\mathfrak{R} = \begin{pmatrix} * & * & * & * \\ 0 & \epsilon & 0 & * \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$
$\dim \mathfrak{R}$ 7	Invariants $\cong C$	$\det_q = 1$
S2b'	$A_{21} = \alpha e_{12} + \alpha e_{24}, \quad \alpha \neq 0$	$\mathfrak{R} = \begin{pmatrix} * & * & 0 & * \\ 0 & \epsilon & 0 & * \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \cong T_3$
$\dim \mathfrak{R}$ 6	Invariants $\cong C \oplus C$ $\mathbf{diag}(\beta, \beta, \delta, \beta)$ $I = (1 - \gamma_0)(1 + i\gamma_{12})$	$\det_q = 1$

G2b'	$A_{11} = \text{diag}(q^2, q, q, 1)$ $A_{12} = e_{12} + e_{24}$ $A_{21} = \alpha A_{12}, \alpha \neq 0$ $A_{22} = A_{11}^{-1} + q^{-1}\alpha e_{14} + \beta e_{33}$ $\beta \neq -q^{-1}, 0$	$\Re = \begin{pmatrix} * & * & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \cong T_3 \oplus C$
$\dim \Re$ 7	Invariants $\cong C \oplus C$ $\text{diag}(\delta, \delta, \epsilon, \delta)$ $I = (1 - \gamma_0)(1 + i\gamma_{12})$	$\det_q = 1 + q\beta e_{33}$
S3	$\mathbf{A}_{11} = \mathbf{diag}(\mathbf{q}^2, \mathbf{q}^2, \mathbf{q}, 1)$ $A_{12} = e_{13} + e_{34}$ $A_{21} = \alpha A_{12}, \alpha \neq 0$ $A_{22} = A_{11}^{-1} + \alpha q^{-1}e_{14}$	$\Re = \begin{pmatrix} \epsilon & 0 & * & * \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cong T_3$
$\dim \Re$ 6	Invariants $\cong T_2$ $\text{diag}(\beta, \gamma, \beta, \beta) + \delta e_{12}$ $I_1 = (1 + \gamma_0)(1 - i\gamma_{12})$ $I_2 = (1 + \gamma_0)(\gamma_1 + i\gamma_2)\gamma_3$	$\det_q = 1$
G3a	$A_{22} = A_{11}^{-1} + \alpha q^{-1}e_{14} + \beta e_{22}$ $\alpha, \beta \neq 0, \beta \neq -q^{-2}$	$\Re = \begin{pmatrix} * & 0 & * & * \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cong T_3 \oplus C$
$\dim \Re$ 7	Invariants $\cong C \oplus C$ $\text{diag}(\gamma, \delta, \gamma, \gamma)$ $I = (1 + \gamma_0)(1 - i\gamma_{12})$	$\det_q = 1 + \beta q^2 e_{22}$
G3b	$A_{22} = A_{11}^{-1} + \alpha q^{-1}e_{14} + e_{12}$ $\alpha \neq 0$	$\Re = \begin{pmatrix} \epsilon & * & * & * \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$
$\dim \Re$ 7	Invariants $\cong T'_2$ $\text{diag}(\beta, \beta, \beta, \beta) + \delta e_{12}$ $I = (1 + \gamma_0)(\gamma_1 + i\gamma_2)\gamma_3$	$\det_q = 1 + e_{12}$

S4a	$\mathbf{A}_{11} = \mathbf{diag}(\mathbf{q}^3, \mathbf{q}^2, \mathbf{q}, 1)$ $A_{12} = e_{12} + e_{23} + e_{34}$ $A_{21} = \alpha A_{12}, \alpha \neq 0$ $A_{22} = A_{11}^{-1} + \alpha q^{-2} e_{13} + \alpha q^{-1} e_{24}$	$\mathfrak{R} = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cong T_4$
$\dim \mathfrak{R}$ 10	Invariants $\cong C$	$\det_q = 1$
S4b	$A_{11} = \text{diag}(q^3, q^2, q, 1)$ $A_{12} = e_{12} + e_{23}$ $A_{21} = \alpha A_{12}, \alpha \neq 0$ $A_{22} = A_{11}^{-1} + \alpha q^{-2} e_{13}$	$\mathfrak{R} = \begin{pmatrix} * & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \cong T_3 \oplus C$
$\dim \mathfrak{R}$ 7	Invariants $\cong C \oplus C$ $\text{diag}(\beta, \beta, \beta, \delta)$ $I = (1 - \gamma_0)(1 - i\gamma_{12})$	$\det_q = 1$
G4b	$A_{11} = \text{diag}(q^3, q^2, q, 1)$ $A_{12} = e_{12} + e_{23}$ $A_{21} = \alpha A_{12}, \alpha \neq 0$ $A_{22} = A_{11}^{-1} + \alpha q^{-2} e_{13} + \beta e_{44}$ $\alpha, \beta \neq 0, \beta \neq -1$	$\mathfrak{R} = \begin{pmatrix} * & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \cong T_3 \oplus C$
$\dim \mathfrak{R}$ 7	Invariants $\cong C \oplus C$ $\text{diag}(\gamma, \gamma, \gamma, \delta)$ $I = (1 - \gamma_0)(1 - i\gamma_{12})$	$\det_q = 1 + \beta e_{44}$

S5	$\mathbf{A}_{11} = \mathbf{diag}(\alpha, \mathbf{q}^2, \mathbf{q}, 1)$ $\alpha \neq 0, q^{-1}, 1, q, q^2$ $A_{12} = e_{23} + e_{34}$ $A_{21} = \beta A_{12}, \beta \neq 0$ $A_{22} = A_{11}^{-1} + \beta q^{-1} e_{24}$	$\mathfrak{R} = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cong T_3 \oplus C$
$\dim \mathfrak{R}$ 7	Invariants $\cong C \oplus C$ $diag(\gamma, \delta, \delta, \delta)$ $I = (1 + \gamma_0)(1 + i\gamma_{12})$	$det_q = 1$
G5	$A_{22} = A_{11}^{-1} + q^{-1}\beta e_{24} + \gamma e_{11}$ $\beta \neq 0, \gamma \neq -\alpha^{-1}, 0$	$\mathfrak{R} = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cong T_3 \oplus C$
$\dim \mathfrak{R}$ 7	Invariants $\cong C \oplus C$ $diag(\delta, \epsilon, \epsilon, \epsilon)$ $I = (1 + \gamma_0)(1 + i\gamma_{12})$	$det_q = 1 + \alpha\gamma e_{11}$
S6	$\mathbf{A}_{11} = \mathbf{diag}(\mathbf{q}^2, \mathbf{q}^2, \mathbf{q}, 1) + \mathbf{e}_{12}$ $A_{12} = e_{13} + e_{34}$ $A_{21} = \alpha A_{12}, \alpha \neq 0$ $A_{22} = A_{11}^{-1} + \alpha q^{-1} e_{14}$	$\mathfrak{R} = \begin{pmatrix} \epsilon & * & * & * \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$
$\dim \mathfrak{R}$ 7	Invariants $\cong T'_2$ $diag(\beta, \beta, \beta, \beta) + \delta e_{12}$ $I = (1 + \gamma_0)(\gamma_1 + i\gamma_2)\gamma_3$	$det_q = 1$
G6	$A_{22} = \begin{pmatrix} q^{-2} & \xi - 1 & 0 & \alpha q^{-1} \\ 0 & q^{-2} & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\mathfrak{R} = \begin{pmatrix} \epsilon & * & * & * \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$
$\dim \mathfrak{R}$ 7	Invariants $\cong T'_2$ $diag(\beta, \beta, \beta, \beta) + \delta e_{12}$ $I = (1 + \gamma_0)(\gamma_1 + i\gamma_2)\gamma_3$	$det_q = 1 + q^2 \xi e_{12}, \xi \neq 0$

S7	$\mathbf{A}_{11} = \mathbf{diag}(\mathbf{q}^2, \mathbf{q}, \mathbf{1}, \mathbf{1}) + \mathbf{e}_{34}$ $A_{12} = e_{12} + e_{24}$ $A_{21} = \alpha A_{12}, \alpha \neq 0$ $A_{22} = A_{11}^{-1} + \alpha q^{-1} e_{14}$	$\Re = \begin{pmatrix} * & * & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & \epsilon & * \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$
$\dim \Re$ 7	Invariants $\cong T'_2$ $diag(\beta, \beta, \beta, \beta) + \delta e_{34}$ $I = (1 - \gamma_0)(\gamma_1 + i\gamma_2)\gamma_3$	$det_q = 1$
G7	$A_{22} = \begin{pmatrix} q^{-2} & 0 & 0 & \alpha q^{-1} \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 1 & \xi - 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\Re = \begin{pmatrix} * & * & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & \epsilon & * \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$
$\dim \Re$ 7	Invariants $\cong T'_2$ $diag(\beta, \beta, \beta, \beta) + \delta e_{34}$ $I = (1 - \gamma_0)(\gamma_1 + i\gamma_2)\gamma_3$	$det_q = 1 + \xi e_{34}, \xi \neq 0$

From the Table, it is easy to see that the following result is valid (roughly speaking this means that the quantum determinants are the only quantum invariants).

Corollary 3 . *If an inner action defined by a representation of $GL_q(1, C)$, $q^m \neq 1$, in $C(1, 3)$ with nontrivial perturbation is not an action of $SL_q(2, C)$, then it has only one basic invariant which can be taken to be equal to the quantum determinant.*

Of course for $SL_q(2, C)$ -actions this result is not valid (in this case the quantum determinant equals 1). Nevertheless, we have seen (formulae (122, 121)) that for every invariant D there exists a connected $GL_q(2, C)$ -action defined by a representation for which the quantum determinant equals D (even if the initial $SL_q(2, C)$ -action is defined by a representation with trivial perturbation). So we can formulate

Corollary 4 . *Every invariant of the inner $SL_q(2, C)$ -action on $C(1, 3)$ is equal to a quantum determinant of a connected $GL_q(2, C)$ -action.*

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